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Optimal interval lengths for nonlocal boundary value problems associated with third order Lipschitz equations

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Abstract

For the third order differential equation, $y''' = f(x, y, y', y'')$, where $f(x, y_1, y_2, y_3)$ is Lipschitz continuous in terms of y_i, $i = 1, 2, 3$, we obtain optimal bounds on the length of intervals on which there exist unique solutions of certain nonlocal three and four point boundary value problems. These bounds are obtained through an application of the Pontryagin Maximum Principle from the theory of optimal control.

Keywords: Nonlinear boundary value problem; Third order Lipschitz equation; Nonlocal boundary condition; Existence; Uniqueness; Optimal control

1. Introduction

In this paper, we shall be concerned with the differential equation

$$y''' = f(t, y, y', y''), \tag{1}$$

for which the assumptions contained in following hypothesis will hold throughout.

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Hypothesis 1.1. $f(t, y_1, y_2, y_3) : \mathcal{D} = (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ is assumed to be continuous and to satisfy a Lipschitz condition of the following form:

$$\left|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)\right| \leqslant \sum_{i=1}^{3} k_i |y_i - z_i|, \tag{2}$$

where the inequality in (2) holds uniformly on $\mathcal{D}$ for the constants k_i, $i = 1, 2, 3$.

In terms of the Lipschitz constants k_i, $i = 1, 2, 3$, we characterize the optimal length for subintervals of (a, b) on which unique solutions exist for boundary value problems consisting of Eq. (1) and either of the *nonlocal* three-point boundary conditions given by

$$y(t_1) = y_1, \qquad y'(t_1) = y_2, \qquad y(t_2) - y(t_3) = y_3; \tag{3}$$

$$y(t_1) - y(t_2) = y_1, \qquad y(t_3) = y_2, \qquad y'(t_3) = y_3, \tag{4}$$

where $a < t_1 < t_2 < t_3 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$, or for boundary value problems consisting of Eq. (1) together with either of the *nonlocal* four-point boundary conditions given by

$$y(t_1) = y_1, \qquad y(t_2) = y_2, \qquad y(t_3) - y(t_4) = y_3; \tag{5}$$

$$y(t_1) - y(t_2) = y_1, \qquad y(t_3) = y_2, \qquad y(t_4) = y_3, \tag{6}$$

where $a < t_1 < t_2 < t_3 < t_4 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$.

Remark 1.2. For the remainder of this paper, we shall denote the boundary value problem consisting of Eq. (1) and boundary condition (j) by (1):(j).

Of particular importance to the development that follows, is the fact that conditions which guarantee uniqueness of solutions for boundary value problems (1):(3) and (1):(4) or (1):(5) and (1):(6) are sufficient to guarantee the existence of solutions for boundary value problems (1):(3), (1):(4), (1):(5), and (1):(6). These facts are a consequence of the following two results from a recent paper by Clark and Henderson [10].

Theorem 1.3 *(Clark–Henderson). Solutions for the boundary value problems* (1):(5) *and* (1):(6) *are unique, when they exist on* (a, b), *if and only if solutions for boundary value problems* (1):(3) *and* (1):(4) *are unique when they exist on* (a, b).

Theorem 1.4 *(Clark–Henderson). If solutions, when they exist, for boundary value problems* (1):(5) *and* (1):(6) *are unique on* (a, b), *then solutions for boundary value problems* (1):(3) *and* (1):(4) *or* (1):(5) *and* (1):(6) *exist and are unique on* (a, b).

Our characterization of the optimal length for subintervals of (a, b) on which unique solutions exist for boundary value problems (1):(3) and (1):(4) or (1):(5) and (1):(6) involves an application of Pontryagin's Maximum Principle to obtain a characterization, in terms of the Lipschitz constants k_i, $i = 1, 2, 3$, of the optimal length for subintervals of (a, b) on which unique solutions exist for boundary value problems consisting of (1) and the two-point focal boundary value problems formed from (1) and

$$y(t_1) = y_1, \qquad y'(t_1) = y_2, \qquad y'(t_2) = y_3; \tag{7}$$

$$y(t_1) = y_1, \qquad y(t_2) = y_2, \qquad y'(t_2) = y_3, \tag{8}$$

where $a < t_1 < t_2 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$. The connection between this characterization and a similar characterization for our three-point and four-point nonlocal problems is explained by the observation that through an application of the Mean Value Theorem to Theorem 1.3 we obtain the following result:

Theorem 1.5. *If solutions for boundary value problems* (1):(7) *and* (1):(8) *are unique, when they exist on* (a, b)*, then solutions for boundary value problems* (1):(3) *and* (1):(4) *are unique when they exist on* (a, b).

Thus, in light of Theorem 1.3, conditions sufficient to guarantee uniqueness of solutions, when they exist on (a, b), for two-point focal boundary value problems (1):(7) and (1):(8), are sufficient to guarantee uniqueness of solutions when they exist on (a, b) for either three-point or four-point nonlocal boundary value problems (1):(3) and (1):(4) or (1):(5) and (1):(6).

The manner in which we apply the Pontryagin Maximum Principle has some history with primary motivation found in the works of Melentsova [30] and Melentsova and Mil'shtein [31, 32]. These works were later adapted to the context of several types of nonlinear boundary value problems by Jackson [24,25], Eloe and Henderson [13], Hankerson and Henderson [19], and Henderson et al. [21–23].

Third order ordinary differential equations have received attention both for their applied as well as theoretical interest. Such equations arise in models for boundary layer theory in fluid mechanics; as for example, when considering convection in a porous medium or a flow adjacent to a standing wall; cf. [1,6,9,12,20,35,36,38,39]. Works for third order equations have also dealt with upper and lower solutions, multiple solutions, nonlinear eigenvalue problems, periodic solutions, monotone boundary conditions, limit point and limit circle criteria, cf. [2,4,7,8,26,33].

Nonlocal boundary value problems also have been of interest both in applications and theory as can be seen in the following papers and the references therein: [3,14,15,17,18,29,34,37]. In particular, for third order nonlocal boundary value problems whose form are closely related to the problems considered in this paper, see those papers by Liu et al. [28], Benbouziane et al. [5], and Du et al. [11].

2. Optimal intervals for uniqueness of solutions

In this section, we apply the Pontryagin Maximum Principle to obtain a characterization, in terms of the Lipschitz constants k_i, $i = 1, 2, 3$, described in (2), for the optimal length of subintervals of (a, b) on which solutions are unique, when they exist, for two-point focal boundary value problems (1):(7) and (1):(8). This length, it will be argued, is optimal for uniqueness of solutions for three-point nonlocal boundary value problems given by (1):(3) and (1):(4), and by Theorem 1.3 for four-point nonlocal boundary value problems (1):(5) and (1):(6). In the course of our arguments, we shall deal primarily with the boundary value problem (1):(7); analogous results can be obtained for the boundary value problem (1):(8) through a transformation by reflection.

Our formulation begins with the definition of a set $\mathcal{U}$ of vector valued *control functions* $\mathbf{v}(t) = (v_1(t), v_2(t), v_3(t))^\top \in \mathbb{R}^3$, whose members have components which satisfy the following conditions for $t \in (a, b)$:

(1) $v_i(t)$, $i = 1, 2, 3$, are Lebesgue measurable;
(2) $|v_i(t)| \leqslant k_i$, $i = 1, 2, 3$.

Our concern will be with boundary value problems associated with linear equations of the form

$$x''' = u_1(t)x + u_2(t)x' + u_3(t)x'', \tag{9}$$

where $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))^\top \in \mathcal{U}$.

If $y(t)$ and $z(t)$ are distinct solutions of the boundary value problem (1):(7) so that their difference $x(t) := y(t) - z(t)$ satisfies

$$x(t_1) = x'(t_1) = x'(t_2) = 0, \tag{10}$$

for some $a < t_1 < t_2 < b$, and if $u_i(t)$, $i = 1, 2, 3$, are defined by

$$u_1(t) = \begin{cases} \frac{f(t,y(t),y'(t),y''(t)) - f(t,z(t),y'(t),y''(t))}{y(t)-z(t)}, & y(t) \neq z(t), \\ 0, & y(t) = z(t), \end{cases}$$

$$u_2(t) = \begin{cases} \frac{f(t,z(t),y'(t),y''(t)) - f(t,z(t),z'(t),y''(t))}{y'(t)-z'(t)}, & y'(t) \neq z'(t), \\ 0, & y'(t) = z'(t), \end{cases}$$

$$u_3(t) = \begin{cases} \frac{f(t,z(t),z'(t),y''(t)) - f(t,z(t),z'(t),z''(t))}{y''(t)-z''(t)}, & y''(t) \neq z''(t), \\ 0, & y''(t) = z''(t), \end{cases}$$

then $u_i(t)$ is Lebesgue measurable, $|u_i(t)| \leqslant k_i$, $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))^\top \in \mathcal{U}$ for $i = 1, 2, 3$, on (a, b), and $x(t)$ is a nontrivial solution of the boundary value problem (9) and (10). From optimal control theory (cf. Gamkrelidze [16, p. 147], or Lee and Markus [27, p. 259]), there is a boundary value problem (9) and (10) which has a nontrivial time optimal solution; that is, there exists at least one nontrivial $\mathbf{u}^* \in \mathcal{U}$ and points $t_1 \leqslant c < d \leqslant t_2$ such that

$$x''' = u_1^*(t)x + u_2^*(t)x' + u_3^*(t)x'', \tag{11}$$

$$x(c) = x'(c) = x'(d) = 0, \tag{12}$$

has a nontrivial solution, $x_0(t)$, and $d - c$ is a minimum over all such solutions. For this time optimal solution, $x_0(t)$, let $\mathbf{x}_0(t) = (x_0(t), x_0'(t), x_0''(t))^\top$. Then, $\mathbf{x}_0(t)$ is a solution of the first order system whose form is given by

$$\mathbf{x}' = -A\big(\mathbf{u}^*(t)\big)^\top \mathbf{x}, \quad t \in (a, b), \tag{13}$$

has a nontrivial solution, $\mathbf{x}^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^\top$ such that for a.e. $t \in [c, d]$,

(i) $\sum_{i=1}^3 x_0^{(i)}(t)x_i^*(t) = \langle \mathbf{x}_0'(t), \mathbf{x}^*(t)\rangle = \max_{\mathbf{u}\in\mathcal{U}}\{\langle A(\mathbf{u}(t))\mathbf{x}_0(t), \mathbf{x}^*(t)\rangle\}$,
(ii) $\langle \mathbf{x}_0'(t), \mathbf{x}^*(t)\rangle$ is a nonnegative constant,
(iii) $x_3^*(c) = x_1^*(d) = x_3^*(d) = 0$.

The maximum condition in (i) can be rewritten as

$$x_3^*(t)\sum_{i=1}^3 u_i^*(t)x_0^{(i-1)}(t) = \max_{\mathbf{u}\in\mathcal{U}}\left\{x_3^*(t)\sum_{i=1}^3 u_i(t)x_0^{(i-1)}(t)\right\} \tag{14}$$

for a.e. $t \in [c, d]$.

Now, by its time optimality and Rolle's theorem, $x_0(t) \neq 0$, $t \in (c, d]$. We may assume without loss of generality, that $x_0(t) > 0$ on $(c, d]$. If $x_3^*(t)$ has no zeros on (c, d), then we can use (14) in determining an optimal control $\mathbf{u}^*(t)$, for a.e. $t \in [c,d]$. We now address the single signature of $x_3^*(t)$ on (c,d).

Toward that end, if $\bar{\mathbf{u}} \in \mathcal{U}$ is such that the boundary value problem given by (9) and (10) for some $a < t_1 < t_2 < b$, has a nontrivial solution, then the adjoint system

$$\alpha' = -A^\top\big(\bar{\mathbf{u}}(t)\big)\alpha, \quad t \in (a,b), \tag{15}$$

$$\alpha_3(t_1) = \alpha_1(t_2) = \alpha_3(t_2) = 0, \tag{16}$$

also has a nontrivial solution, and conversely. Hence the Pontryagin Maximum Principle associates with a time optimal solution of boundary value problem (9):(10) a time optimal solution of boundary value problem (15):(16), and conversely. Hence, it follows by its own time optimality that $x_3^*(t)$ does not vanish on (c,d).

Since $x_0(t) > 0$ on $(c,d]$, we have from (14) that if $x_3^*(t) < 0$ on (c,d), then the time optimal solution $x_0(t)$ is a solution of

$$x''' = -k_1x - k_2|x'| - k_3|x''| \tag{17}$$

on $[c,d]$, whereas if $x_3^*(t) > 0$ on (c,d), then the time optimal solution, $x_0(t)$, is a solution of

$$x''' = k_1x + k_2|x'| + k_3|x''| \tag{18}$$

on $[c,d]$.

Our discussion thus far is based on the premise that (1) has distinct solutions whose difference satisfies (10). We now see that if the appropriate sign conditions are satisfied by the optimal solution $x_0(t)$ of boundary value problem (9):(10) and by the component $x_3^*(t)$ of the solution of the associated adjoint system (13), optimal intervals can be determined on which only trivial solutions exist for boundary value problems (10):(17) or (10):(18). And as a consequence, solutions of the boundary value problem (1):(7) will be unique on such subintervals.

Theorem 2.1. *If there is a vector-valued function* $\mathbf{u}$*, such that* $\mathbf{u}(t) \in \mathcal{U}$ *for all* $t \in (a,b)$*, for which the boundary value problem* (9):(10) *has a nontrivial solution for some* $a < t_1 < t_2 < b$*, and if* $x_0(t)$ *is a time optimal solution satisfying* (12) *where* $d - c$ *is a minimum, then* $x_0(t)$ *is a solution of* (17) *on* $[c,d]$.

Proof. We have already observed in the preceding discussion that $x_0(t)$ is a solution of (17) or (18) on $[c,d]$. We may assume without loss of generality that $x_0''(c) > 0$ so that indeed $x_0(t) > 0$ on $(c,d]$. If $\mathbf{x}^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^\top$ is a nontrivial solution of the adjoint system (13) associated with $x_0(t)$, then by the Pontryagin Maximum Principle, $x_3^*(c) = x_1^*(d) = x_3^*(d) = 0$, and by its time optimality $x_3^*(t) \neq 0$ on (c,d). From the nature of Eqs. (17) or (18), $x_0'''(t)$ is of one sign on (c,d), and so $x_0''(t)$ is strictly monotone on $[c,d]$. From the assumption that $x_0''(c) > 0$ and the boundary conditions $x_0(c) = x_0'(c) = x_0'(d) = 0$, it follows that $x_0'''(t) < 0$ on (c,d), and as a consequence, that $x_0(t)$ is a solution of (17) on $[c,d]$. $\square$

By a parallel development and a change of variable with respect to solutions of (9), we can also establish a dual result.

Theorem 2.2. *If there is a* $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t))^\top \in \mathcal{U}$*, such that the corresponding linear equation* (9) *has a nontrivial solution satisfying*

$$x'(t_1) = x(t_2) = x'(t_2) = 0, \tag{19}$$

for some $a < t_1 < t_2 < b$, then there exists at least one nontrivial $\mathbf{u}^* \in \mathcal{U}$ and points $t_1 \leqslant \sigma < \gamma \leqslant t_2$ such that the boundary value problem consisting of (11) *and*

$$x'(\sigma) = x(\gamma) = x'(\gamma) = 0,$$

has a nontrivial solution $w_0(t)$ such that $\gamma - \sigma$ is a minimum. Moreover, $w_0(t)$ is a solution of (18) *on* $[c, d]$.

Remark 2.3. We make an important observation at this junction. In part, because both (17) and (18) are autonomous, there is a relationship between $x_0(t)$ in Theorem 2.1 and $w_0(t)$ in Theorem 2.2. In particular, a straightforward argument yields that $w_0(t)$ is a translation followed by a reflection of $x_0(t)$; in which case, it necessarily follows that $d - c = \gamma - \sigma$.

Theorem 2.4. *Let $\ell = \ell(k_1, k_2, k_3) > 0$ be the smallest positive number such that there exists a solution $x(t)$ of the boundary value problem*

$$\begin{aligned} &x''' = -k_1 x - k_2 x' - k_3 |x''|, \\ &x(0) = x'(0) = x'(\ell) = 0, \end{aligned} \tag{20}$$

with $x(t) > 0$ on $(0, \ell]$, or $\ell = \infty$ if no such solution exists. If $y(t)$ and $z(t)$ are distinct solutions of the boundary value problems (1):(7) *and* (1):(8) *for some $a < t_1 < t_2 < b$, and if $t_2 - t_1 < \ell$, it follows that $y(t) \equiv z(t)$ on $[t_1, t_2]$, and this is best possible for the class of all differential equations satisfying the Lipschitz condition given in* (2).

Proof. Since Eqs. (17) and (18) are autonomous, it suffices to apply Theorem 2.1 (and Theorem 2.2 relative to the remark) with respect to the boundary conditions at 0 and ℓ.

First, if there are distinct solutions $y(t)$ and $z(t)$ of (1) whose difference $w(t) = y(t) - z(t)$ satisfies (10), where $t_2 - t_1 < \ell$, then $w(t)$ is a nontrivial solution of the boundary value problem (9):(10), for appropriately defined $\mathbf{u} \in \mathcal{U}$. Then, from the discussion and Theorem 2.1, Eq. (17) has a nontrivial solution on a subinterval of length less than ℓ. But, by the minimality of ℓ, such a boundary value problem can have only the trivial solution: a contradiction. Therefore the solutions of the boundary value problem (1):(7) are unique whenever $t_2 - t_1 < \ell$.

Second, if there exist distinct solutions $\bar{y}(t)$ and $\bar{z}(t)$ of (1) whose difference $\bar{w}(t) = \bar{y}(t) - \bar{z}(t)$ satisfies (19) where $t_2 - t_1 < \ell$, then $\bar{w}(t)$ is a nontrivial solution of boundary value problem (9):(19) for appropriate $\bar{u} \in \mathcal{U}$. Application of Theorem 2.2 and the remark following Theorem 2.2 yield the same contradiction as above. So the solutions of the boundary value problem (1):(8) are unique whenever $t_2 - t_1 < \ell$.

This is best possible from the fact that both (17) and (18) satisfy the Lipschitz condition (2), and if $\ell \neq \infty$, then $x(t)$ is a nontrivial solution of (17) and (10) on $[0, \ell]$. The boundary value problem also has the trivial solution. $\square$

Because of the uniqueness relations stated in Theorem 1.5, we can apply the Theorem 2.4 to obtain optimal intervals for uniqueness of solutions of boundary value problems (1):(3) and (1):(4).

Theorem 2.5. *Let ℓ be as in Theorem* 2.4. *If $y(t)$ and $z(t)$ are distinct solutions for the boundary value problems* (1):(3) *and* (1):(4), *for some $a < t_1 < t_2 < t_3 < b$, and if $t_3 - t_1 \leqslant \ell$, it follows that $y(t) \equiv z(t)$ on $[t_1, t_3]$, and this is best possible for the class of all differential equations satisfying the Lipschitz condition* (2).

Proof. In view of Theorems 1.5 and 2.4, solutions of the boundary value problems (1):(3) and (1):(4) are unique when $t_3 - t_1 \leqslant \ell$. To see again that this is best possible, consider the solution $x(t)$ in Theorem 2.4. This is a nontrivial solution of (17) taking the form (20).

Let $\epsilon > 0$ be sufficiently small that $x(t)$ is a solution of (20) on $[0, \ell + \epsilon]$. Now $x'''(t) < 0$ on $[0, \ell + \epsilon]$, and it follows from the boundary conditions satisfied by $x(t)$ that $x'(\ell) = 0$ and that $x''(\ell) < 0$. In particular, $x(t)$ has a positive maximum at ℓ. So there exist $0 < \tau_1 < \ell < \tau_2 < \ell + \epsilon$ such that $x(t)$ is a nontrivial solution of (17) satisfying $x(0) = x'(0) = x(\tau_1) - x(\tau_2) = 0$. This boundary value problem also has the trivial solution. Since $\epsilon > 0$ was arbitrary, the "best possible" statement follows for uniqueness of solutions of the boundary value problem (1):(3). Verification with respect to the uniqueness of solutions of the boundary value problem (1):(4) follows as in previous arguments. □

3. Optimal intervals for existence of solutions

In this section, we make a simple application of Theorem 2.5 in conjunction with the uniqueness implies existence result for nonlocal boundary values found in Theorem 1.4. We then follow this with an example for the case when $k_1 = k_2 = k_3 = 1$.

Theorem 3.1. *Let ℓ be as in Theorem* 2.5. *Then, the boundary value problems* (1):(5) *and* (1):(6) *have a unique solution, provided that $t_4 - t_1 < \ell$. In addition, the boundary value problems* (1):(3) *and* (1):(4) *have a unique solution, provided that $t_3 - t_1 < \ell$. Moreover, this result is best possible for the class of all third order ordinary differential equations* (1) *satisfying the Lipschitz condition found in* (2).

Example 3.2. In this example, when $k_1 = k_2 = k_3 = 1$, we compute the optimal interval length ℓ for which there exist unique solutions for the boundary value problems (1):(3), (1):(4), (1):(5) and (1):(6) on subintervals whose length is no more than ℓ.

In particular, let $x(t)$ be the solution of

$$x''' - x - x' - |x''|,$$
$$x(0) = x'(0) = 0, \qquad x''(0) = 1,$$

and let $\eta > 0$ be the first positive number such that $x'(\eta) = 0$. Then, $\eta = \ell$ of Theorem 2.5, and we find by elementary methods that $\eta = \ell = 1.94766$. This may be stated as a concluding result.

Theorem 3.3. *Suppose* (1) *satisfies the Lipschitz condition,*

$$\bigl|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)\bigr| \leqslant \sum_{i=1}^{3} |y_i - z_i|. \tag{21}$$

Then, the boundary value problems (1):(5) *and* (1):(6) *have unique solutions, provided that $t_4 - t_1 < 1.94766$. In addition, each of the boundary value problems* (1):(3) *and* (1):(4) *have unique solutions, provided that $t_3 - t_1 < 1.94766$. Moreover, this result is best possible for the class of all third order ordinary differential equations* (1) *satisfying the Lipschitz condition* (21).

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